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Equivalence of massless boson and fermion theories in curved two-dimensional space–time: Sugawara stress tensor

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Abstract. It is shown that the well known equivalence of two-dimensional massless scalar and spinor quantum field theories extends to curved space–time. The vacuum Sugawara stress tensor for a spinor field is evaluated by a covariant point-separation procedure and found to be identical to the scalar vacuum stress, previously evaluated by Davies and Unruh to be equivalent to the conventional fermion stress tensor (in the absence of compact spatial sections).

1. Equivalence of fermion and boson theories

It has been noticed (see for example Freundlich 1972) that in two-dimensional infinite Minkowski space there is a close formal correspondence between the quantum field theories of free massless fermions and bosons. It has long been known (Jordan 1935) that it is possible to construct massless bosons from bilinear combinations of massless fermion fields, while the reverse construction has also been carried out (Skyrme 1961). Moreover, Coleman *et al* (1969) have shown that the theory of free massless fermions in two dimensions is a Sugawara theory. In such a theory one deals directly with commutators of the current operator $j_\mu(x)$, defined by a point-splitting procedure.

$$j_\mu(x)=\lim_{\epsilon\to 0}\bar{\Psi}(x+\epsilon)\gamma_\mu\Psi(x-\epsilon) \tag{1.1}$$

(spinor indices will be suppressed). The stress tensor is defined in terms of the current j_μ as follows†

$$\bar{T}_{\mu\nu}(x)=\lim_{\epsilon\to 0}\pi(j_\mu(x)j_\nu(x)+j_\nu(x)j_\mu(x)-g_{\mu\nu}(x)j^\sigma(x)j_\sigma(x)). \tag{1.2}$$

It is clear that the Sugawara stress tensor $\bar{T}_{\mu\nu}$ is structurally very different from the conventional fermion stress tensor $T_{\mu\nu}$ (given by (2.8) below). However, $\bar{T}_{\mu\nu}$ closely resembles the scalar stress tensor

$$\tfrac{1}{2}(\phi_{,\mu}\phi_{,\nu}+\phi_{,\nu}\phi_{,\mu}-g_{\mu\nu}\phi^{,\sigma}\phi_{,\sigma}) \tag{1.3}$$

through the replacement $j_\mu=\phi_{,\mu}$. In fact it has been shown (Freundlich and Lurié

† Definition (1.2) differs by a factor of 2 from that used conventionally. This is because we use four- rather than two-component spinors.

1970, Freundlich 1970) that this Sugawara model is equivalent to a canonical Goldstone boson theory with the stress tensor derived from a Lagrangian containing only the field ϕ.

It is is interesting to investigate to what extent this equivalence remains true in curved two-dimensional space. In this case the fields are no longer free, but are coupled to an external metric ('gravitational') field. As a first step, we here investigate the vacuum expectation value of $\bar{T}_{\mu\nu}$ in curved space and prove that it is indeed identical to the scalar field case, so long as the Casimir energy is absent (no discrete modes).

2. The model

A general two-dimensional space–time may be described by the metric

$$ds^2 = C(u, v)\,du\,dv \tag{2.1}$$

where u and v are standard retarded and advanced null coordinates respectively: $u \equiv t - z$, $v \equiv t + z$.

The neutrino field $\Psi(u, v)$ obeys the equations

$$\gamma^\mu \nabla_\mu \Psi = 0 \tag{2.2}$$

$$(1 - i\gamma^5)\Psi = 0 \tag{2.3}$$

where $\nabla_\mu \equiv \partial_\mu + \Gamma_\mu$ is the covariant derivative, Γ_μ being the affine spinor connection (spinor indices will be suppressed throughout). We use the following representation of the Dirac matrices (see Unruh 1974):

$$\gamma_u = C^{1/2}\gamma_b, \qquad \gamma_v = C^{1/2}\gamma_a \tag{2.4}$$

$$\gamma_a = \tfrac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}, \qquad \gamma_b = \tfrac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \tag{2.5}$$

$$\gamma^0 = \tfrac{1}{2}(\gamma^u + \gamma^v) = C^{-1/2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2.6}$$

$$\gamma^5 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}. \tag{2.7}$$

The stress tensor operator is

$$\begin{aligned} T_{uu} &= \tfrac{1}{4}\mathrm{i}[\bar{\Psi}, \gamma_u \nabla_u \Psi] + \text{HC} \\ T_{vv} &= \tfrac{1}{4}\mathrm{i}[\bar{\Psi}, \gamma_v \nabla_v \Psi] + \text{HC} \\ T_{uv} &= T_{vu} = 0 \end{aligned} \tag{2.8}$$

where $\bar{\Psi}$ is the Dirac adjoint of Ψ, HC denotes Hermitian conjugate and [,] is the commutator.

Both (2.2) and (2.8) are invariant under conformal transformations $g_{\mu\nu} \to C g_{\mu\nu}$, provided that we also transform Ψ to $C^{-1/4}\Psi$ and $\bar{\Psi}$ to $C^{-1/4}\bar{\Psi}$. Therefore, we may exploit the conformally flat nature of the two-dimensional metric (2.1) to solve equations (2.2) and (2.3) immediately in terms of standard normalised exponential mode functions

$$\psi_\omega(t, x) = \begin{cases} (8\pi)^{-1/2} C^{-1/4} \begin{bmatrix} 1 \\ 0 \\ k/\omega \\ 0 \end{bmatrix} \exp[-\mathrm{i}(kz - \omega t)] & (2.9) \\ (8\pi)^{-1/2} C^{-1/4} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -k/\omega \end{bmatrix} \exp[-\mathrm{i}(kz + \omega t)] & (2.10) \end{cases}$$

with $\omega = |k| > 0$ and $-\infty < k < \infty$. These two positive-frequency solutions represent neutrinos with both helicities. Two more linearly independent solutions denoted by $\psi_{-\omega}$ describe the antineutrinos ($\omega < 0$), but we shall not bother to consider these explicitly. By symmetry, the antineutrinos merely contribute an overall factor of 2 to the stress tensor.

As usual, the field operator Ψ may be expanded in terms of a complete set of mode functions of the form ψ_ω:

$$\Psi = \int_{-\infty}^{\infty} \mathrm{d}k \, (c_\omega \psi_\omega + d_\omega^\dagger \psi_{-\omega}). \tag{2.11}$$

The operators c_ω, d_ω are annihilation operators for neutrinos and antineutrinos respectively. The positive-frequency (neutrino) term in (2.11) may be written, with the help of (2.9) and (2.10) as

$$(4\pi)^{-1/2} C^{-1/4} \int_0^\infty \mathrm{d}\omega \sum_{r=1}^{2} c_\omega^{(r)} u_\omega^{(r)} \mathrm{e}^{-\mathrm{i}\omega u} + v \text{ term} \tag{2.12}$$

where

$$u_\omega^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u_\omega^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

The superscripts 1 and 2 represent the two helicity states, $u_\omega^{(1)}$ and $u_\omega^{(2)}$ being solutions of $(1 \mp i\gamma^5)\ u_\omega = 0$ respectively. The v term is identical to the first term with $e^{-i\omega u}$ replaced by $e^{-i\omega v}$, and the spinors $u_\omega^{(1)}$and $u_\omega^{(2)}$ replaced by

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

respectively. The first term of (2.12) represents neutrinos moving to the right, the second term neutrinos moving to the left. In the latter case the helicities are the reverse of the former. Only the first term contributes to T_{uu}. By spatial symmetry, T_{vv} will be an identical expression, with u replaced by v. We need therefore only calculate T_{uu} explicitly.

The annihilation operators c_ω and d_ω define a vacuum state $|0\rangle$ through the requirement

$$c_\omega|0\rangle = d_\omega|0\rangle = 0. \tag{2.13}$$

3. Construction of the Sugawara stress tensor

The generalisation of (1.1) to include the effect of external fields has long been known. In the electromagnetic case, it is necessary to introduce factors

$$\exp\left(ie\int_x^{x+\epsilon} A_\sigma(x')\,dx^\sigma\right) \tag{3.1}$$

into the definition of the current operator to preserve gauge invariance. In the 'gravitational' case, general covariance can be preserved by the analogous expression

$$\exp\left(\int_x^{x(+\epsilon)} \Gamma_\sigma(x')\,dx^\sigma\right) \tag{3.2}$$

where Γ_σ is the spin affine connection which obeys the equations

$$\begin{aligned} &\partial_\nu\gamma_\mu - \Gamma^\rho_{\mu\nu}\gamma_\rho + \gamma_\mu\Gamma_\nu - \Gamma_\nu\gamma_\mu = 0 \\ &\mathrm{Tr}\,\Gamma_\mu = 0. \end{aligned} \tag{3.3}$$

The factor (3.2) has the effect of parallel transporting spinors from the point $x(+\epsilon)$ to x, where $x(+\epsilon)$ is displaced a proper distance ϵ from x along a geodesic passing through x with (normalised) tangent vector at x of $t^\sigma(\epsilon)$, which may be seen by noting that (3.2) is a solution of the spinor parallel transport equation

$$\begin{aligned} &\frac{dS}{d\epsilon} + S\Gamma_\sigma t^\sigma(\epsilon) = 0 \\ &S(0) = \mathbf{1} \end{aligned} \tag{3.4}$$

where $\mathbf{1}$ is the unit matrix and $t^\sigma(\epsilon) \equiv dx^\sigma/d\epsilon$.

Equation (3.4) may easily be solved as a power series in ϵ to yield

$$S(\epsilon) = \alpha\gamma_a\gamma_b + \alpha^{-1}\gamma_b\gamma_a \tag{3.5}$$

where

$$\alpha=\left(\frac{C(\epsilon)}{C(0)}\right)^{1/4} U^{1/2}(\epsilon) \qquad \text{and} \qquad U(\epsilon)=\frac{t^u(\epsilon)}{t^u(0)}.$$

In what follows we shall denote quantities dependent on x evaluated at $x(\pm\epsilon)$ with a $\pm$ subscript or superscript respectively. The curved space generalisation of (1.1) may be taken to be

$$j_\mu(x)=\lim_{\epsilon\to 0} \overline{S_+\Psi_+}\,\gamma_\mu(x)S_-\Psi_-. \tag{3.6}$$

Substituting (3.6) into the expression for $\bar{T}_{\mu\nu}$ in (1.2) leads to three terms involving products of four spinors, of the general form

$$\overline{S_+\Psi_+}S_-\Psi_-\overline{S_+\Psi_+}S_-\Psi_-. \tag{3.7}$$

Wick's theorem may be used to evaluate the vacuum expectation value of expression (3.7). The result is

$$-G(x_\epsilon-x_{-\epsilon})G(x_\epsilon-x_{-\epsilon}) \tag{3.8}$$

where $G(x, x')$ is the bispinor defined by

$$G(x, x')=\mathrm{i}\{S_-\Psi^{(+)}(x), \overline{S_+\Psi^{(-)}(x')}\}. \tag{3.9}$$

The $(\pm)$ superscripts denote positive- and negative-frequency parts of Ψ respectively and $\{\ ,\ \}$ denotes the anticommutator.

It follows from (1.2) and (3.8) that

$$\begin{aligned}\langle 0|\bar{T}_{\mu\nu}(x)|0\rangle = -\pi \lim_{\epsilon\to 0} \mathrm{Tr}(&\gamma_\mu(x)G(x_\epsilon-x_{-\epsilon})\gamma_\nu(x)G(x_\epsilon-x_{-\epsilon})\\ &+\gamma_\nu(x)G(x_\epsilon-x_{-\epsilon})\gamma_\mu(x)G(x_\epsilon-x_{-\epsilon})\\ &-g_{\mu\nu}(x)g^{\sigma\rho}(x)\gamma_\sigma(x)G(x_\epsilon-x_{-\epsilon})\gamma_\rho(x)G(x_\epsilon-x_{-\epsilon})).\end{aligned} \tag{3.10}$$

Thus the Sugawara vacuum stress tensor is constructed from essentially the square of the bispinor G.

It is now necessary to evaluate G explicitly. To this end we use the mode functions (2.12) in the definition (3.9). This gives

$$G(x_\epsilon-x_{-\epsilon})=\mathrm{i}(4\pi)^{-1}(C_+C_-)^{-1/4}\int_{-\infty}^{\infty}\mathrm{d}k\sum_{r=1}^{2} S_-u_\omega^{(r)}S_+\bar{u}_\omega^{(r)}\exp[\mathrm{i}k(z-z')-\mathrm{i}|k|(t-t')] \tag{3.11}$$

when the anticommutation relations $\{c_\omega^{(r)}, c_{\omega'}^{(r')}\}=\delta_{\omega\omega'}\delta_{rr'}$ are used.

It is easy to prove the following relation:

$$\sum_{r=1}^{2}(S_+u_\omega^{(r)})(S_-\bar{u}_\omega^{(r)})=\begin{cases}\alpha_+\alpha_-\gamma_a, & k>0\\ (\alpha_+\alpha_-)^{-1}\gamma_b, & k<0.\end{cases} \tag{3.12}$$

Moreover, we note from the normalisation condition $Ct^ut^v=1$ that $(U_+U_-)^{-1}=V_+V_-C_+C_-C^{-2}$. Then, carrying out the integral in (3.11), yields

$$G(x_\epsilon-x_{-\epsilon})=-(4\pi)^{-1}C^{-1/2}(x)\left(\frac{(U_+U_-)^{1/2}}{\Delta u}\gamma_a+\frac{(V_+V_-)^{+1/2}}{\Delta v}\gamma_b\right), \tag{3.13}$$

where $\Delta u=u'-u$, $\Delta v=v'-v$ and $u'\equiv u(\epsilon)$, $u\equiv u(-\epsilon)$, etc.

This expression may now be used in (3.10) to give terms of the general type

$$-(16\pi)^{-1}C^{-1}\,\mathrm{Tr}\left(\frac{U_+U_-}{(\Delta u)^2}\gamma_\mu\gamma_a\gamma_\nu\gamma_b+\frac{(V_+V_-)^{+1}}{(\Delta v)^2}\gamma_\mu\gamma_b\gamma_\nu\gamma_b\right). \tag{3.14}$$

The two cross terms containing $(\Delta u\,\Delta v)^{-1}$ vanish identically because $\gamma_\mu\gamma_a\gamma_\nu\gamma_b=\gamma_\mu\gamma_b\gamma_\nu\gamma_a=0$ for $\mu,\ \nu=u,\ v$.

For the $u-u$ component of $\langle 0|\bar{T}_{\mu\nu}|0\rangle$, the second term under the trace sign in (3.14) vanishes because it contains the factor $\gamma_u\gamma_b\gamma_u\gamma_b\propto\gamma_b^4=0$. Similarly, for the $v-v$ component, the first term vanishes. The remaining traces are easily evaluated:

$$\mathrm{Tr}(\gamma_u\gamma_a\gamma_u\gamma_a)=C\,\mathrm{Tr}(\gamma_b\gamma_a\gamma_b\gamma_a)=C\,\mathrm{Tr}(\gamma_b\gamma_a)=2C$$

and

$$\mathrm{Tr}(\gamma_v\gamma_b\gamma_v\gamma_b)=2C.$$

Expression (3.14) then reduces to

$$-\frac{U_+U_-}{8\pi(\Delta u)^2},\qquad -\frac{V_+V_-}{8\pi(\Delta v)^2} \tag{3.15}$$

for the $u-u$ and $v-v$ components respectively.

So, combining all three of the terms of the form (3.15) finally yields

$$\begin{aligned}\langle 0|\bar{T}_{uu}(x;\epsilon,t^\sigma)|0\rangle&=-\frac{1}{4\pi}U_+U_-(\Delta u)^{-2}\\ \langle 0|\bar{T}_{vv}(x;\epsilon,t^\sigma)|0\rangle&=-\frac{1}{4\pi}V_+V_-(\Delta v)^{-2}\\ \langle 0|\bar{T}_{uv}(x;\epsilon,t^\sigma)|0\rangle&=\langle 0|\bar{T}_{vu}(x;\epsilon,t^\sigma)|0\rangle=0.\end{aligned} \tag{3.16}$$

Equations (3.16) are exactly the same as were obtained for the massless scalar field (compare for example equation (2.27) of Davies and Fulling 1977). Thus we have shown that, in two dimensions, even for a general metric (i.e. even in the presence of an external 'gravitational' field) the massless spinor quantum field may be used to construct a massless boson quantum theory using the Sugawara stress tensor, with the natural generalisation of the current operator j_μ to curved space–time (expression (3.6)), at least as far as the vacuum expectation values are concerned. The extension of this result to the operators themselves depends on some subtleties concerning point-splitting. We hope to deal with this whole problem more fully elsewhere.

Because the *regularised* scalar vacuum stress tensor is identical to that for neutrinos in two dimensions (Davies and Unruh 1977), it follows that the Sugawara tensor agrees with the conventional neutrino stress tensor (2.8) for this quantity. This is a rather surprising result in view of their totally different structure. This result is no longer true if there is a Casimir term (discrete modes), because this enters into the two expressions with opposite signs.

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